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Cauchy problem for gauge field theories with external sources

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Abstract. The initial-value problem is studied for two models of gauge field theory in the presence of arbitrary external sources. The non-Abelian SU(2) Yang-Mills theory with a vanishing external charge $J_0^a = 0$ allows for a clear distinction of the gauge field degrees of freedom into dynamical and non-dynamical ones. The case of general external sources J_μ^a introduces new dynamical quantities—Lagrange multipliers Q^a. Thus a modified Lagrangian has been taken as a starting point. A similar analysis is carried out for an Abelian model of scalar electrodynamics. In all three cases time evolution along classical equations of motion imposes no restriction on external sources and the dynamical degrees of freedom can take almost arbitrary initial values.

1. Introduction

The present status of gauge field theories seems to be twofold. On the one hand, it is accepted that they form (or can form) a field theoretical basis of the strong, weak and electromagnetic interactions [1] while, on the other hand, the calculation apparatus is still inadequate to describe the whole physics of these phenomena. One usually uses perturbation methods which are appropriate when couplings are weak†, and even in some more sophisticated approaches as the expansion around solitons [3] one does not depart from semiclassical approximation schemes. In these methods gauge fields can be studied either on the classical or the quantum level only after a choice of some gauge condition has been made. However, this introduces new problems. For example, in a non-Abelian case the covariant gauge condition leads to such complications as an appearance of the Faddeev-Popov ghosts [4] or the Gribov ambiguity [5].

Actually those obstacles can be overcome with the help of external sources which are kept non-zero everywhere. The mere presence of the gauge non-invariant source terms removes singularities connected with gauge freedom. Thus either the path integral over all trajectories can be evaluated [6] or the canonical path-integral quantisation can be carried out unambiguously [7].

Different aspects of gauge field theories coupled to external sources has been already studied by several authors (for the latest references, see [8]). However, in most cases, the domain of non-zero sources has been restricted to either finite regions or separate points. We stress that in the source-free regions all problems mentioned earlier reappear. Thus we will consider external sources which vanish nowhere (with a possible exception at the spacetime infinity). This makes things look different.

† In the lattice formulation of gauge field theories the strong coupling regime has been developed [2]. However, its continuous limit seems to be neither reliable nor straightforward.

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We are able to find a system of consistent equations of motion for dynamical degrees of freedom for the classical SU(2) Yang-Mills theory. This can be accomplished by the help of a proper parametrisation of gauge fields A_i^a. This way the Cauchy problem for the non-Abelian gauge fields with arbitrary external sources will have a unique solution if some solvability conditions are satisfied. These conditions which are to be imposed on the gauge field potential are not very restrictive and should not be confused with any *non-linear* or *implicit* external current conservation law. We stress that there are no dynamically induced restrictions on external sources; even more, we find our analysis easier when they are of generic form.

When the external charge J_0^a is different from zero, then some new degrees of freedom become dynamical. They can be expressed as the Lagrange multiplier fields Q^a entering the Lagrange density and this leads to the modified equations of motion. The dynamical degrees of freedom are found easily and their initial-value problem can be solved unambiguously. Furthermore, in some class of possible initial values, it is shown that this modified system is equivalent to the naive Yang-Mills one.

Our paper is organised as follows. First we will consider a simple case of external sources arbitrary current J_i^a and will solve the Cauchy problem for the SU(2) Yang-Mills fields coupled to such a source. Then we will turn to the generic case of J_μ^a and will carry out the same analysis. Finally our results will be discussed and some still unsolved problems will be pointed out. In the appendices we will sketch some proofs omitted throughout the main presentation and we will analyse an Abelian model of gauge field theory—the scalar electrodynamics with external sources.

2. The SU(2) Yang-Mills fields coupled to external currents J_i^a

First we would like to explain our motivations for the choice of the case presented. The SU(2) Yang-Mills field theory is a simple system which possesses all basic ingredients of other non-Abelian models. Also, there is a possibility of introducing a convenient parametrisation of gauge field potentials A_μ^a and external currents J_μ^a. Thus we are able to classify arbitrary J_μ^a in terms of the gauge group and the Lorentz group invariants. We can also divide arbitrary A_μ^a into dynamical and constrained degrees of freedom.

For the sake of clarity of our presentation we will reach our goal in two steps: first, we will explain our way of reasoning in a simpler case when $J_0^a=0$, and later we will attack a generic case.

The system under consideration is described by the Lagrangian density:

$$L=-\tfrac{1}{4}F^{a\mu\nu}F_{\mu\nu}^a-A_i^aJ_i^a \tag{1}$$

where

$$F_{\mu\nu}^a=\partial_\mu A_\nu^a-\partial_\nu A_\mu^a+g\varepsilon^{abc}A_\mu^bA_\nu^c.$$

Now we can use variational principles in order to arrive at the Lagrange-Euler equations of motion:

$$D_\mu^{ab}F^{b\mu i}-J^{ai}=0 \tag{2a}$$

$$D_i^{ab}F_{i0}^b=0 \tag{2b}$$

where:

$$D_\mu^{ab}=\delta^{ab}\partial_\mu+g\varepsilon^{acb}A_\mu^c.$$

In the vacuum case $J_i^a = 0$, equations (2a) and (2b) form a starting point for the dynamical analysis. However, due to a gauge symmetry freedom those equations are singular—the second time derivative of A_0^a does not appear at all. Usually this difficulty is cured by a gauge condition imposed on the dynamical A_i^a fields. Then a value of A_0 is given by equation (2b).

For non-zero external current we have a different situation—the consistency (integrability) condition of (2a) and (2b) is no longer trivial:

$$D_i^{ab} J_i^b = 0. \tag{3}$$

Hitherto this condition has been interpreted as a kind of '*generalised*' (*non-Abelian*) *current conservation law* [8], e.g. only those currents J_i^a which fulfil (3) could be consistently coupled to the Yang–Mills system. This reasoning originates either in the Abelian counterpart of (3):

$$\partial_i J_i = 0$$

which no doubt constrains J_i, or in an expansion in powers of the coupling constant g. However we heavily stress that (3) should be treated non-perturbatively and then things will change considerably.

First we introduce a proper description of arbitrary external currents $J_i^a(x)$ by means of two symmetrical matrices:

$$K^{ab}(x) := J_i^a(x) J_i^b(x) \tag{4a}$$

$$L_{ij}(x) := J_i^a(x) J_j^a(x). \tag{4b}$$

Evidently the matrix K and its eigenvalues are the O(3) group invariants while L and its eigenvalues are the SU(2) group invariants. Furthermore, those eigenvalues coincide (for the proof see appendix 1). Thus they are simultaneously gauge and rotationally invariant.

We may classify the possible J_i^a into three categories:

(i) all eigenvalues of K (and L) are non-zero,

(ii) one eigenvalue is zero,

(iii) two eigenvalues vanish.

The first two categories describe an external current which is intrinsically non-Abelian, while the third one corresponds to an Abelian current. We stress that this distinction is a local one. Thus it is possible that some $J_i^a(x)$ fall into different categories in different regions of spacetime.

In the following analysis we will suppose that J_i^a belongs to the first category. Thus the following parametrisation of the gauge field potential is possible:

$$A_i^a = J_j^b \Delta_{ij} (S^{ab} + \varepsilon^{abc} N^c) \tag{5}$$

where $L_{ij}\Delta_{jk} = \Delta_{ij}L_{jk} = \delta_{ik}$, $S^{ab} = S^{ba}$.

Now if we plug this formula into (3) then we will easily find that (for details see appendix 1)

$$N^a = -(1/2g)\partial_i J_i^a. \tag{6}$$

This way three degrees of freedom of gauge field potential are 'frozen' by the covariant conservation law (3). In contrast, the symmetric matrix S^{ab} is unconstrained and dynamical.

We can rewrite our equations of motion (2*a*) in terms of S^{ab} and N^a. For the sake of clarity we will not go into all details here—we will just show what kind of dynamics is described by them. Thus we will be interested in the leading terms (with $\ddot{A}_i$ and $\dot{A}_0$) while the other terms will be omitted as non-leading ones and marked by dots. In such an approximation (2*a*) is

$$\ddot{A}_i^a - D_i^{ab}\dot{A}_0^b + \ldots = 0 \tag{2a'}$$

and after introducing (5) it gives two separate equations for S^{ab} and N^a:

$$2\ddot{S}^{ab} - (J_i^a\partial_i\dot{A}_0^b + J_i^b\partial_i\dot{A}_0^a) - g[\varepsilon^{adc}(S^{db} + \varepsilon^{dbe}N^e) + \varepsilon^{bdc}(S^{da} + \varepsilon^{dae}N^e)]\dot{A}_0^c + \ldots = 0 \tag{7a}$$

$$2\ddot{N}^a - \varepsilon^{abc}J_i^b\partial_i\dot{A}_0^c - g(S^{bb}\delta^{ac} - S^{ac} + \varepsilon^{acb}N^b)\dot{A}_0^c + \ldots = 0. \tag{7b}$$

Furthermore from equation (6) we find that $\ddot{N}^a$ is no longer a leading term. Thus we should omit it in our leading derivative approximation. In such a way we obtain a *dynamical equation* for A_0^a:

$$M^{ab}\dot{A}_0^b + \ldots = 0 \tag{8}$$

where M^{ab} is a linear differential operator:

$$M^{ab} := \varepsilon^{abc}J_i^c\partial_i + g(S^{cc}\delta^{ab} - S^{ab} + (1/2g)\varepsilon^{abc}\partial_i J_i^c). \tag{8'}$$

Now we will anticipate our future discussion of the invertibility (non-singularity) of M and say that we can solve (8) for $\dot{A}_0^a$. Accordingly we are in a position to impose the Cauchy problem. Let us suppose that, at the initial moment of time (at the spacelike surface) $t = t_0$, the initial values of S^{ab}, $\dot{S}^{ab}$, A_0^a are given by arbitrary C^2 functions (of spatial coordinates). In order to reproduce these quantities at any later moment $t > t_0$ we must have some prescription for $\ddot{S}^{ab}$ and $\dot{A}_0$. Now it presents no difficulty, because we can read them from (7*a*) and (8), respectively. However, these equations may be inconsistent—we should find their conditions of integrability.

First we observe that

$$D_i^{ab}D_\mu^{bc}F^{c\mu i} \equiv D_0^{ab}D_i^{bc}F_{i0}^c \tag{9}$$

which is another form of a well known identity:

$$D_\mu^{ab}D_\nu^{bc}F^{c\mu\nu} \equiv 0.$$

Then, from the dynamical equations for S^{ab} and A_0^a (2) and the constraint for N^a (3) we find that the LHS of (*g*) vanishes. Thus we conclude that

$$D_0^{ab}G^b = \dot{G}^a + g\varepsilon^{abc}A_0^bG^c = 0 \tag{10}$$

where

$$G^a := D_i^{ab}F_{i0}^b = (D_i^2)^{ab}A_0^b - D_i^{ab}\dot{A}_i^b. \tag{10'}$$

This equation is the integrability condition for our basic equations (2*a*) and (3). Due to its form (of a dynamical equation for G^a), it will not hinder our previous analysis at all. We see that there are two equivalent possibilities for the evaluation of $G^a(t)$ for arbitrary t. Firstly, we can calculate $G^a(t_0)$ by means of (10′) and then evolve it according to (10). Secondly, we can use the prescription (10′) for any t. In any case we must evaluate the truly dynamical quantities S^{ab}, $\dot{S}^{ab}$, A^a. Thus, from the practical point of view the first routine seems to be useless. However, in some cases it can give very promising results. For example, if G^a vanishes at t_0 then it will vanish at any later moment $t>t_0$ if only A_0^a remains regular during the period (t_0, t). This property is a local one—it concerns a particular point x. Thus, if we *impose* that G^a vanishes everywhere at t_0 then we will *obtain* that it vanishes everywhere at any later $t>t_0$.

The imposition of any condition on initial values of dynamical variables diminishes a number of the independent initial degrees of freedom. In our case we may interpret the condition $G^a(t_0)=0$ as an equation for $A_0^a(t_0)$:

$$(D_i^2)^{ab}A_0^b = D_i^{ab}\dot{A}_i^b. \tag{11}$$

Now we have two differential (in three-dimensional space) equations (8) and (11) which should be solved. First let us suppose that at every point of spacetime the third spatial coordinate of external current J_i^a can be transformed by a regular gauge transformation into the form

$$J_3^a = J_3\delta^{a3}. \tag{12}$$

This supposition is valid since we have limited our discussion to category (i) of external currents for any smooth J_3^a.

Further let us take some spacelike surface $z=z_0$ and impose the following conditions for A_0^a:

$$A_0^a(x, y, z_0, t) = \varphi^a(x, y, t) \qquad a=1,2 \tag{13a}$$

$$A_0^3(x, y, z_0, t_0) = \varphi^3(x, y). \tag{13b}$$

At this surface (8) becomes

$$J_3\partial_3\dot{A}_0^2 + g(S^{cc}-S^{11})\dot{\varphi}^1 + g(-S^{12}+(1/2g)\partial_iJ_i^3)\dot{\varphi}^2$$
$$+g(-S^{13}-(1/2g)\partial_iJ_i^2)\dot{A}_0^3+\ldots=0 \tag{14a}$$

$$-J_3\partial_3\dot{A}_0^1 + g(S^{cc}-S^{22})\dot{\varphi}^2 + g(-S^{12}-(1/2g)\partial_iJ_i^3)\dot{\varphi}^1$$
$$+g(-S^{23}+(1/2g)\partial_iJ_i^1)\dot{A}_0^3+\ldots=0 \tag{14b}$$

$$g(S^{cc}-S^{33})\dot{A}_0^3 + g(-S^{13}+(1/2g)\partial_iJ_i^2)\dot{\varphi}^1 + g(-S^{23}-(1/2g)\partial_iJ_i^1)\dot{\varphi}^2+\ldots=0. \tag{14c}$$

We have left non-leading terms containing $\dot{\varphi}^1$ and $\dot{\varphi}^2$ in order to visualise the symmetry of these equations. This symmetry shows that at (12) we could specify a different direction in colour space as well.

Now we may employ (14c) to find $\dot{A}_0^3(x, y, z_0, t)$ if only the following condition is satisfied at the surface z_0:

$$S^{11}+S^{22}\neq 0. \tag{15}$$

We may substitute this quantity into (14a) and (14b) in order to evaluate $\partial_3\dot{A}_0^2$ and $\partial_3\dot{A}_0^1$, respectively. These derivatives and (13a) will allow us to calculate $\dot{A}_0^1$ and $\dot{A}_0^2$ on the surface

$$z = z+\Delta z$$

where Δz is infinitesimally small. At this new surface the whole procedure may be repeated if the condition (15) is satisfied. Thus we see that, if (15) is satisfied everywhere, then (8) may be solved unambiguously.

Furthermore we would like to make a similar 'recurrence' analysis of another differential equation (11). We plan to solve it at an initial time t_0. Thus we may determine the values of $A_0^a(x, y, z_0, t_0)$ from (13a) and (13b). However, this will not do enough here because we should also know the values of

$$\partial_3 A_0^a(x, y, z_0, t_0) = \chi^a(x, y). \tag{16}$$

Now we may write (11) in a simplified form:

$$(\partial_3^2 \delta^{ab} + 2g\varepsilon^{acb} A_3^c \partial_3) A_0^b + \ldots = 0 \tag{11'}$$

where, exceptionally, dots stand for terms other than $\partial_3 A_0^a$ and $\partial_3^2 A_0^a$. Clearly the recurrence procedure can be carried out only if all species will stay regular—no special restrictions are to be put on them.

Accordingly, $A_0^a(x, y, z, t_0)$ and $\dot{A}_0^a(x, y, z, t_0)$ can be evaluated from (11) and (8), respectively. We want to remind ourselves that equation (11) will be automatically satisfied for any later $t > t_0$ if the system evolves according to the dynamical equations (2a) and the constraint (3). Thus we are not forced to solve (11) for any time t and (13a), (13b) and (16) form a set of sufficient boundary conditions.

We now turn to the truly dynamical quantities and find that they need initial-value conditions

$$S^{ab}(x, y, z, t_0) = \eta^{ab}(x, y, z) \tag{17a}$$

$$\dot{S}^{ab}(x, y, z, t_0) = \lambda^{ab}(x, y, z). \tag{17b}$$

At last we may formulate a final conclusion that, *if we take (17a) and (17b) as the initial conditions for S^{ab} and (13a) and (13b) and (16) as the boundary conditions for A_0^a, then our dynamical system governed by equations (2a) and (3) is equivalent to a system governed by primary (Lagrange–Euler) equations (2a) and (2b).* We stress that we have no limitations on J_i^a. On the contrary, we need it in a general non-singular form.

3. Generic external currents J_μ^a

In the previous section we have limited arbitrary external currents to those with a zero external charge $J_0^a = 0$. Now we would like to release this restriction. However, the case of generic J_μ^a brings new complications. Evidently the antisymmetric product

$$\varepsilon_{\mu\nu\lambda\rho} J_\mu^a J_\nu^b J_\lambda^c J_\rho^d \equiv 0 \tag{18}$$

vanishes identically for the SU(2) gauge group because there are only three directions in the colour space. Thus we may define a Lorentz vector n_μ:

$$n_\mu := \varepsilon^{bcd} \varepsilon_{\mu\nu\lambda\rho} J_\nu^b J_\lambda^c J_\rho^d \tag{19a}$$

which is orthogonal to J_μ^a:

$$n_\mu J^{a\mu} = 0. \tag{19b}$$

A colour-invariant symmetric matrix $L_{\mu\nu}$:

$$L_{\mu\nu} := J_\mu^a J_\nu^a \tag{20a}$$

has a zero eigenvalue with n_μ as an eigenvector:

$$L_{\mu\nu} n_\nu = 0. \tag{20b}$$

Accordingly, $L_{\mu\nu}$ is not invertible and we cannot naively generalise the parametrisation (5) by changing three-dimensional space indices to four-dimensional ones. In fact, an analogous parametrisation can be done. However, in our presentation it is of no help.

At any spacetime point, due to a 'constraint' relation (19*b*) for generic J^a_μ we may always take three of them as independent colour vectors. Thus in some regions we may find that J^a_3 vanishes and in other regions we may encounter $J^a_2 = 0$. This possibility presents a serious problem because no Lorentz transformation can bring these two local conditions into a global one. In such a case we should analyse those two regions separately. However, there is an interesting case of generic J^a_μ which we can study without the complications mentioned. We may suppose that, in spite of the non-zero external charge J^a_0, the external current J^a_i is intrinsically non-Abelian—it belongs to category (i).

Now we propose to make a small modification of our starting point and we will take the following Lagrangian density:

$$L = -\tfrac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} + A^a_\mu J^{a\mu} + Q^a D^{ab}_\mu J^{b\mu} \tag{21}$$

where Q^a plays the role of the Lagrange multipliers.

We can easily arrive at the Lagrange-Euler equations of motion:

$$D^{ab\mu}F^b_{\mu i} + J^a_i - g\varepsilon^{abc}Q^b J^c_i = 0 \tag{22a}$$

$$D^{ab}_i F^b_{i0} - J^a_0 + g\varepsilon^{abc}Q^b J^c_0 = 0 \tag{22b}$$

$$D^{ab}_\mu J^{b\mu} = 0. \tag{22c}$$

Furthermore, we should find a consistency condition for the above system of differential equations—this can be done in a few lines and the result is

$$\varepsilon^{abc}J^{b\mu}D^{cd}_\mu Q^d = 0. \tag{23}$$

We see that, if the Lagrange multipliers Q^a are identically zero, then our system of Lagrange-Euler equations (22*a*) and (22*b*) would be equivalent to the naive Yang-Mills equations. However, as it will turn out later, if we keep Q^a fields as independent degrees of freedom the analysis will be simpler, especially telling dynamical degrees of freedom from non-dynamical ones. We expect that the naive Yang-Mills Lagrangian is not correct for the case of arbitrary external currents if the external charge is non-zero. For example, a canonical path integral quantisation leads to a singular and Lorentz non-invariant result [7]. Accordingly we will not refer to naive equations any longer and we will treat (22*a*), (22*b*) and (22*c*) as *the physical primary equations of motion.*

In order to integrate consistently the primary equations of motion we have to solve their integrability condition (23) simultaneously. Due to (22*c*) we may rewrite (23) in an equivalent form:

$$D^{ab}_0(\varepsilon^{bcd}J^c_0 Q^d) = D^{ab}_i(\varepsilon^{bcd}J^c_i Q^d). \tag{24a}$$

Thus one part of Q^a (orthogonal to J^a_0) is dynamical, while the other one (parallel to J^a_0) is constrained by the equation

$$\varepsilon^{abc}J^a_0 J^b_i D^{cd}_i Q^d = 0. \tag{24b}$$

This observation is important because it shows the possibility of non-zero solutions for Q^a.

Before we proceed further let us suppose that the external charge is a smooth function which can be transformed to the form

$$J_0^a = \rho\delta^{a3} \tag{25}$$

by means of a regular gauge transformation (rotation in colour space). Due to our previous discussion the spatial coordinates of external current belong to category (i). Thus again we may parametrise A_i^a by S^{ab} and N^a:

$$A_i^a = J_j^b \Delta_{ij}(S^{ab} + \varepsilon^{abc} N^c). \tag{5}$$

Now we take the following equations as the independent 'equations of motion':

$$D^{ab\mu} F_{\mu i}^b + J_i^a - g\varepsilon^{abc} Q^b J_i^c = 0 \tag{22a}$$

$$D_\mu^{ab} J^{b\mu} = 0 \tag{22c}$$

$$\varepsilon^{abc} J^{b\mu} D_\mu^{cd} Q^d = 0. \tag{23}$$

First we will analyse the constraint equations (22c) and notice that A_0^1 and A_0^2 are given by

$$A_0^1 = -1/\rho\ (2N^2 - 1/g\partial_i J_i^2) \tag{26a}$$

$$A_0^2 = 1/\rho\ (2N^1 - 1/g\partial_i J_i^1) \tag{26b}$$

for non-zero external charge, while N^3 is constrained by

$$N^3 = (1/2g)(\rho - \partial_i J_i^3). \tag{26c}$$

There is another constraint which we should solve—(24b). Due to our choice of J_0^a (25) and to the parametrisation (5), this equation has the following form:

$$(S^{11} + S^{22})Q^3 = -(S^{13} + N^2)Q^1 - (S^{23} - N^1)Q^2 + 1/g(J_i^2\partial_i Q^1 - J_i^1\partial_i Q^2). \tag{24b'}$$

Thus, if the condition

$$S^{11} + S^{22} \neq 0 \tag{27}$$

is satisfied at every point, then we will be able to express Q^3 in terms of dynamical fields.

Dynamical equations for Q^1 and Q^2 can be easily obtained from (23):

$$\begin{aligned} &\rho[\dot{Q}^1 + g(A_0^2 Q^3 - A_0^3 Q^2)] \\ &\quad = g(S^{11} + S^{33})Q^2 + g(S^{12} - N^3)Q^1 + g(S^{23} + N^1)Q^3 \\ &\quad\quad + J_i^3\partial_i Q^1 - J_i^1\partial_i Q^3 \end{aligned} \tag{28a}$$

$$\begin{aligned} &-\rho[\dot{Q}^2 + g(A_0^3 Q^1 - A_0^1 Q^3)] \\ &\quad = g(S^{22} + S^{33})Q^1 + g(S^{12} + N^3)Q^1 + g(S^{13} - N^2)Q^3 \\ &\quad\quad + J_i^2\partial_i Q^3 - J_i^3\partial_i Q^2. \end{aligned} \tag{28b}$$

We have left all terms written down explicitly in order to visualise how another choice of J_0^a would change this procedure into a different one. Always, those directions in the colour space which are orthogonal to J_0^a will be dynamical, while those parallel to J_0^a will be constrained. This proves our earlier statement based on equations (24a) and (24b).

In further analysis we will use our leading derivative approximation again. Thus, from (22*a*) we will obtain dynamical equations for S^{ab}, N^1 and N^2:

$$2\ddot{S}^{ab}-(J_i^a\delta^{b3}+J_i^b\delta^{a3})\partial_i\dot{A}_0^3-g[\varepsilon^{ad3}(S^{db}+\varepsilon^{dbe}N^e)+\varepsilon^{bd3}(S^{da}+\varepsilon^{dae}N^e)]\dot{A}_0^3+\ldots=0 \qquad (29a)$$

$$2\ddot{N}^a-\varepsilon^{ab3}J_i^b\partial_i\dot{A}_0^3-g(S^{bb}\delta^{a3}-S^{a3}+\varepsilon^{a3b}N^b)\dot{A}_0^3+\ldots=0 \qquad \text{for } a=1,2. \qquad (29b)$$

Due to the constraint (26*c*), $\ddot{N}^3$ is not a leading term and the appropriate equation gains a form

$$g(S^{11}+S^{22})\dot{A}_0^3+\ldots=0 \qquad (29c)$$

and we have a dynamical equation for A_0^3 only if the previous condition (27) is met everywhere.

Accordingly, we have found true equations of motion for any unconstrained quantity. However our system of equations (22*a*), (22*c*) and (23) can be integrated consistently if their integrability condition is satisfied for every t:

$$D_0^{ab}G'^b=0 \qquad (30)$$

where G'^a stands for

$$G'^a:=D_i^{ab}F_{i0}^b-\rho\delta^{a3}+g\rho\varepsilon^{ab3}Q^b=(D_i^2)^{ab}A_0^b-D_i^{ab}\dot{A}_i^b-\rho\delta^{a3}+g\rho\varepsilon^{ab3}Q^b. \qquad (30a)$$

Here we have the possibility to keep G'^a identically equal to zero everywhere only if we force it to vanish at an initial surface $t=t_0$. First we plan to calculate A_0^3 from G'^a:

$$[\partial_i^2-g^2(A_i^1A_i^1+A_i^2A_i^2)]A_0^3+\ldots=0. \qquad (31a)$$

Thus we have to impose some boundary conditions. For example, if we take

$$A_0^3(x,y,z_0,t_0)=\varphi(x,y) \qquad (32a)$$

$$\partial_3A_0^3(x,y,z_0,t_0)=\eta(x,y) \qquad (32b)$$

then (31*a*) may be integrated for the surface $t=t_0$. Further, from G'^2 and G'^1 we may evaluate Q^1 and Q^2, respectively:

$$Q^1=(1/g\rho)((D_i^2)^{2b}A_0^b-D_i^{2b}\dot{A}_i^b) \qquad (31b)$$

$$Q^2=-(1/g\rho)((D_i^2)^{1b}A_0^b-D_i^{1b}\dot{A}_i^b). \qquad (31c)$$

Thus we are at the end of our analysis and we may formulate the Cauchy problem. *At the initial surface $t=t_0$, dynamical quantities S^{ab}, N^1, N^2, $\dot{S}^{ab}$, $\dot{N}^1$, $\dot{N}^2$ are given by arbitrary C^2 functions of spatial coordinates. The boundary conditions (32a) and (32b) imposed on A_0^3 allow for its evaluation at t_0, while Q^1 and Q^2 can be calculated from (31b) and (31c). All those quantities are to be evolved for any $t>t_0$ by means of appropriate equations of motion (29a), (29b), (29c), (28a) and (28b). The non-dynamical degrees of freedom A_0^1, A_0^2, N^3, Q^3 should be calculated from (26a), (26b), (26c) and (24b′). Such a dynamical system will be equivalent to that governed by the primary Lagrange–Euler equations of motion (22a), (22b) and (22c).*

4. Conclusions

In this paper we have presented a detailed analysis of the Cauchy problem for gauge field theories coupled to external sources. Our aim was to formulate this problem unambiguously for the general case of external sources.

We have succeeded in two cases of external currents coupled to the non-Abelian SU(2) Yang–Mills theory and for the Abelian scalar electrodynamics (see appendix 2). We have encountered no conservation law or any other limitation on external sources but we have found some conditions on the field variables. Fortunately, they are not very restrictive and the dynamical fields can take almost arbitrary initial values. Still, a time evaluation of such systems can break down at some future time—thus our solution is only a local one.

We can argue that our identification of dynamical equations of motion is self consistent by making a general observation.

All dynamical degrees of freedom fall into two categories—they evolve either according to second-order (elliptic) or first-order equations of motion. The existence and uniqueness of solution for the time period (t_0, t) *will come about according to the usual arguments only if the solvability conditions mentioned before will be met in that period of time.*

Here we would like to answer the inevitable objection that we have to solve a plethora of mathematical problems before stating that the Cauchy problem is properly formulated. We agree that the mathematical aspects of our reformulation of the initial-value problem may be non-trivial. However, the physical picture of the Cauchy problem is now clear.

Further we would like to mention one important consequence of our considerations, namely that *any adequate expansion scheme should impose no restriction on external currents at any level of approximation.* In a future paper we will present a strong coupling expansion for the SU(2) Yang–Mills theory [9] which meets the above requirement.

In any case we have treated 'a non-linear conservation law for external sources' as a kind of gauge fixing condition induced by the mere presence of such sources. In the case of non-zero non-Abelian external charge, we have found that the Lagrange multipliers can develop non-zero values.

Those unexpected features of gauge fields in the presence of non-trivial external sources can have some physical consequences. We stress that they are invisible by means of the 'brute force' or perturbative methods used so far in an exploration of gauge field theories.

Finally we would like to express our strong conviction that similar analysis can also be carried out for other gauge groups like SU(3), SU(5) etc, though a different parametrisation may be useful.

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Appendix 1. On invariant characterisation of J_i^a and parametrisation of A_i^a

We propose to analyse arbitrary external currents J_i^a in terms of two symmetric matrices:

$$K^{ab} = J_i^a J_i^b \tag{4a}$$

$$L_{ij} = J_i^a J_j^a \tag{4b}$$

arguing that their coinciding eigenvalues are both rotationally and gauge invariant. Here we would like to prove this statement.

First, from the definitions (4a) and (4b) we derive the following equalities:

$$\mathrm{tr}\,(K^{ab}) = \mathrm{tr}\,(L_{ij}) \tag{A1.1a}$$

$$\mathrm{tr}\,(K^{ab}K^{bc}) = \mathrm{tr}\,(L_{ij}L_{jk}) \tag{A1.1b}$$

$$\mathrm{tr}\,(K^{ab}K^{bc}K^{cd}) = \mathrm{tr}\,(L_{ij}L_{jk}L_{km}). \tag{A1.1c}$$

Of course, one can write down equalities of traces for higher powers of K^{ab} and L_{ij}. However, for our purpose the ones given above will be enough.

Let us denote eigenvalues of K^{ab} by $\varkappa_1, \varkappa_2, \varkappa_3$, and those of L_{ij} by $\lambda_1, \lambda_2, \lambda_3$. Now equations (A1.1$a$)-(A1.1$c$) look like

$$\varkappa_1+\varkappa_2+\varkappa_3=\lambda_1+\lambda_2+\lambda_3 \tag{A1.1a'}$$

$$\varkappa_1^2+\varkappa_2^2+\varkappa_3^2=\lambda_1^2+\lambda_2^2+\lambda_3^2 \tag{A1.1b'}$$

$$\varkappa_1^3+\varkappa_2^3+\varkappa_3^3=\lambda_1^3+\lambda_2^3+\lambda_3^3. \tag{A1.1c'}$$

Next we introduce two useful functions:

$$f(z) := (\varkappa_1 - z)(\varkappa_2 - z)(\varkappa_3 - z) \tag{A1.2a}$$

$$g(z) := (\lambda_1 - z)(\lambda_2 - z)(\lambda_3 - z). \tag{A1.2b}$$

By a simple calculation one may find that $f(z)$ can be expressed as

$$\begin{aligned} f(z) &= \varkappa_1\varkappa_2\varkappa_3 - z(\varkappa_1\varkappa_2+\varkappa_1\varkappa_3+\varkappa_2\varkappa_3)+z^2(\varkappa_1+\varkappa_2+\varkappa_3)-z^3 \\ &= \tfrac{1}{6}[(\varkappa_1+\varkappa_2+\varkappa_3)^3+2(\varkappa_1^2+\varkappa_2^2+\varkappa_3^2)-3(\varkappa_1^2+\varkappa_2^2+\varkappa_3^2)(\varkappa_1+\varkappa_2+\varkappa_3)] \\ &\quad + z/2[\varkappa_1^2+\varkappa_2^2+\varkappa_3^2-(\varkappa_1+\varkappa_2+\varkappa_3)^2]+z^2(\varkappa_1+\varkappa_2+\varkappa_3)-z^3. \end{aligned} \tag{A1.3}$$

A similar expression should be written for $g(z)$: $\varkappa_i$ is to be exchanged by λ_i only. Thus, from (A1.1a')-(A1.1c') we notice that those functions are equal:

$$f(z) \equiv g(z). \tag{A1.4}$$

Accordingly, whenever z is equal to some $\varkappa_i$ then $f(\varkappa_i)=0$ but due to (A1.4) $g(\varkappa_i)=0$ also. However, from (A1.2b) we see that $g(z)$ vanishes only at $z=\lambda_i$. Thus we must conclude that

$$\varkappa_i = \lambda_{P(i)} \tag{A1.5}$$

where $P(i)$ stands for some permutation of i.

Further, from (4a) one sees that $\varkappa_i$ is rotationally invariant, while from (4b) one sees that λ_i is gauge invariant. Now due to equation (A1.5), one concludes that these eigenvalues are both rotationally and gauge invariant.

We may define a class of external currents J_i^a by specifying the number of non-zero eigenvalues. If we take the case when all eigenvalues are non-vanishing then

$$\det\,(L_{ij}) \neq 0 \tag{A1.6}$$

and evidently we may define a reciprocal matrix Δ_{ij}:

$$L_{ij}\Delta_{jk} = \delta_{ik}. \tag{A1.7}$$

This allows us to introduce the following parametrisation for any gauge field A_i^a:

$$\begin{aligned} A_i^a &= A_j^a\delta_{ji} = A_j^a L_{jk}\Delta_{ki} = A_j^a J_j^b J_k^b \Delta_{ki} \\ &= (S^{ab} + \varepsilon^{abc}N^c)J_k^b\Delta_{ki} \end{aligned} \tag{A1.8}$$

where

$$S^{ab}=\tfrac{1}{2}(A_i^a J_i^b+A_i^b J_i^a)$$
$$N^a=\tfrac{1}{2}\varepsilon^{abc}A_i^b J_i^c.$$

In order to prove (6) we need to show that a colour matrix X^{ab}:

$$J_i^a\Delta_{ij}J_j^b=X^{ab} \tag{A1.9}$$

is an identity matrix. Let us multiply X^{ab} by K^{bc} and use (A1.7):

$$X^{ab}K^{bc}=J_i^a\Delta_{ij}J_j^bJ_k^bJ_k^c=J_i^a\Delta_{ij}L_{jk}J_k^c=J_i^aJ_i^c.$$

Thus

$$X^{ab}K^{bc}=K^{ac}. \tag{A1.10}$$

Further, due to the non-singularity of K^{ab}:

$$\det(K^{ab})=\det(L_{ij})\neq 0$$

we conclude that

$$X^{ab}=\delta^{ab}. \tag{A1.11}$$

Appendix 2. Scalar electrodynamics with external sources

To broaden our view on general features of the gauge fields with external sources we would like to present here an Abelian model. Our aim is to show that, if theory is non-linear, then arbitrary external sources can be coupled unambiguously and the Cauchy problem can be properly formulated. We propose to consider a system of complex scalar fields ϕ, ϕ^* and vector field A_μ with the Lagrangian density

$$L=-\tfrac{1}{4}f_{\mu\nu}f^{\mu\nu}+|D_\mu\phi|^2+F[|\phi|^2]+\phi^*\eta+\phi\eta^*+A_\mu J^\mu \tag{A2.1}$$

where $f_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$, $D_\mu=\partial_\mu+\mathrm{i}eA_\mu$ and F is some polynomial (quadratic for a renormalisable model). We suppose that J_μ, η and η^* are given by arbitrary smooth functions and we demand that $|\eta|^2$ is non-zero everywhere.

From (A2.1) we may easily obtain appropriate Lagrange–Euler equations of motion:

$$\partial^\mu f_{\mu\nu}-\mathrm{i}e(\phi^*D_\nu\phi-\phi(D_\nu\phi)^*)+J_\nu=0 \tag{A2.2a}$$

$$(-D^\mu D_\mu+2F')\phi+\eta=0 \tag{A2.2b}$$

$$(-D^\mu D_\mu+2F')^*\phi^*+\eta^*=0. \tag{A2.2c}$$

Those equations have a consistency condition:

$$\mathrm{i}e(\phi^*\eta-\phi\eta^*)+\partial^\mu J_\mu=0. \tag{A2.3}$$

Thus we see that some part of the scalar degrees of freedom is constrained. In order to solve this equation explicitly we introduce the following parametrisation:

$$\phi=(2)^{-1/2}\,\mathrm{e}^{i\vartheta}(\alpha+i\beta) \tag{A2.4a}$$

$$\phi^*=(2)^{-1/2}\,\mathrm{e}^{-i\vartheta}(\alpha-\mathrm{i}\beta) \tag{A2.4b}$$

$$A_\mu=B_\varepsilon-1/e\,\partial_\mu\vartheta \tag{A2.4c}$$

where ϑ is the phase factor of scalar sources:

$$\eta = e^{i\vartheta} s \tag{A2.5a}$$

$$\eta^* = e^{-i\vartheta} s. \tag{A2.5b}$$

Introducing these new field variables into (A2.2*a*)-(A2.2*c*) we obtain a new form of Lagrange-Euler equations:

$$\partial^\mu b_{\mu i} + e^2 B_i(\alpha^2+\beta^2) - e\beta \overleftrightarrow{\partial}_i \alpha + J_i = 0 \tag{A2.6a}$$

$$-\partial_i b_{i0} + e^2 B_0(\alpha^2+\beta^2) - e\beta \overleftrightarrow{\partial}_0 \alpha + J_0 = 0 \tag{A2.6b}$$

$$(-\partial^2 + e^2 B^2 + 2F')\beta - 2eB^\mu \partial_\mu \alpha - e\partial_\mu B^\mu \alpha = 0 \tag{A2.6c}$$

$$(-\partial^2 + e^2 B^2 + 2F')\alpha + 2eB^\mu \partial_\mu \beta + e\partial_\mu B^\mu \beta + s = 0 \tag{A2.6d}$$

where

$$A\overleftrightarrow{\partial}_\mu B = A\partial_\mu B - B\partial_\mu A \qquad b_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \qquad \partial^2 = \partial^\nu \partial_\nu.$$

Now the consistency condition (A2.3) has a simple form:

$$\partial_\mu J^\mu + es\beta = 0 \tag{A2.7}$$

and we see that our parametrisation of a scalar sector by means of two real fields is very convenient because equation (A2.7) can be solved immediately:

$$\beta = \beta_0 := -\frac{\partial^\mu J_\mu}{es}. \tag{A2.7$'$}$$

Accordingly the usually dynamical scalar field: β becomes 'nailed to' a definite function β_0 due to the presence of non-zero external source s. Of course, if s vanishes somewhere than (A2.7) degenerates to a true current conservation law:

$$\partial_\mu J^\mu = 0. \tag{A2.8}$$

In our further analysis we suppose that $s \neq 0$ everywhere. Thus β_0 is a well defined function.

Because β is not a dynamical quantity so equation (A2.6*c*) should be interpreted as the dynamical equation for B_0:

$$e\alpha \dot{B}_0 = -2eB_0\dot{\alpha} + (-\partial^2 + e^2B^2 + 2F')\beta_0 - e\alpha\partial_i B_i - 2eB_i\partial_i\alpha \tag{A2.9}$$

if only the dynamical scalar field α is non-zero.

Now we turn to the Cauchy problem and take (A2.6*a*), (A2.6*d*) and (A2.9) as the equations of motion for dynamical fields B_i, B_0 and α. Further, if β is given by a smooth function (A2.7$'$) then our system can be integrated consistently if the following condition holds:

$$\partial_0 g = 0 \tag{A2.10}$$

where

$$g = -\partial_i b_{i0} + e^2 B_0(\alpha^2 + \beta_0^2) - e\beta_0 \overleftrightarrow{\partial}_0 \alpha + J_0. \tag{A2.10$'$}$$

Thus g takes a constant value during the time evolution and we may make it vanish only if, at the initial surface $t = t_0$, B_0 is given by

$$(-\partial_i^2 + e^2\alpha^2 + e^2\beta_0^2) B_0 = -e\beta_0 \overleftrightarrow{\partial}_0 \alpha + J_0. \tag{A2.11}$$

However, solving for B_0 involves some boundary conditions, so we propose to take at a spatial surface $z = z_0$:

$$B(x, y, z_0, t_0) = \chi(x, y) \tag{A2.12a}$$

$$\partial_3 B(x, y, z_0, t_0) = \gamma(x, y). \tag{A2.12b}$$

In such a manner we have arrived at a dynamical system which is equivalent to that given by the primary Lagrange-Euler equations (A2.6*a*)-(A2.6*d*).

Finally we may formulate the Cauchy problem in the following way.

(ii) B_i, $\dot{B}_i$, α, $\dot{\alpha}$ are given as arbitrary C^2 functions at the initial surface $t = t_0$. Additionally, the dynamical scalar field α must be non-vanishing.

(ii) B_0 is given by (A2.11) at t_0.

(iii) The time evolution is governed by (A2.6*a*), (A2.6*d*) and (A2.9).

(iv) The non-dynamical scalar field β is given everywhere by (A2.7′).

This system will evolve smoothly until α vanishes somewhere. Thus we conclude that, also in the Abelian case, the proposed solution is local.

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